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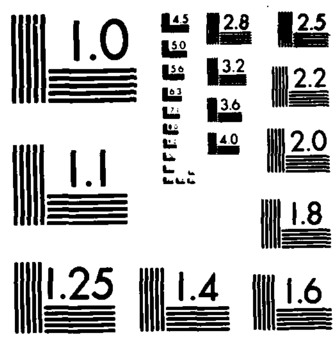

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APPROXIMATE AND LOCAL LINEARIZABILITY OF NONLINEAR DISCRETE-TIME SYSTEMS

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Abstract

We consider a single-input nonlinear discrete-time system of the form

$$\Sigma: \; x(t+1) = f(x(t),u(t))$$

where $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, and $f(x,u): \mathbb{R}^{N+1} \to \mathbb{R}^N$ is a C^∞ $\mathbb{R}^N$-valued function. Necessary and sufficient conditions for approximate linearizability are given for Σ. We also give a sufficient condition for local linearizability. Finally, we present analogous results for multi-input nonlinear discrete-time systems.

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We consider a single-input nonlinear discrete-time system of the form

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where $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, and $f(x,u): \mathbb{R}^{N+1} \to \mathbb{R}^N$ is a C^∞ $\mathbb{R}^N$-valued function. Necessary and sufficient conditions for approximate linearizability are given for Σ. We also give a sufficient condition for local linearizability. Finally, we present analogous results for multi-input nonlinear discrete-time systems.

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I. Introduction

This document ~~We~~ consider a single-input nonlinear discrete-time system of ~~the~~ form

$$\Sigma: x(t+1) = f(x(t),u(t)) \tag{1}$$

where $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, and $f(x,u)$: $\mathbb{R}^{N+1} \to \mathbb{R}^N$ is a C^∞ $\mathbb{R}^N$-valued function.

Many authors have studied (local or global) linearization (Cheng et. al. 1985, Hunt and Su 1981, Jakubczyk and Respondek 1980, Krener 1973, Su 1982) and approximate linearization (Krener 1984) by state feedback and coordinate change for nonlinear continuous-time systems. ~~In~~ this paper ~~we~~ discusses necessary conditions and sufficient conditions for local linearization and approximate linearization by state feedback and coordinate change for nonlinear discrete-time systems. Other related work on nonlinear discrete-time systems can be found in (Grizzle 1985a, 1985b, Grizzle and Nijmeijer 1985, Monaco and Normand-Cyrot 1983a, 1983b). Keywords: matrices (mathematics).

<u>Definition 1</u>: A point (x_e,u_e) such that $f(x_e,u_e) = x_e$ is called an equilibrium point.

Now consider the following linear discrete-time system Σ_0.

$$\Sigma_0: y(t+1) = Ay(t) + bv(t) = g(y(t),v(t)),$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} \quad (N\times N \text{ matrix})$$

$$b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \quad (N\times 1 \text{ matrix})$$

Similar to the continuous-time case (Krener 1984, Su 1982), we can define local linearizability and approximate linearizability for a discrete-time system. Let (x_e,u_e) be an equilibrium point of Σ.

Definition 2: Σ is said to be locally linearizable at (x_e, u_e) if there exist an open neighborhood $U(\subset \mathbb{R}^{N+1})$ of (x_e, u_e) and a diffeomorphism $T: U \to T(U)$ such that (i) $\bar{T} = (T_1, T_2, \ldots, T_N)$ are functions of $x_1, x_2, \ldots, x_N$ only, (ii) $T(x_e, u_e) = 0_{(N+1)\times 1}$, and (iii) $\bar{T} \circ f = g \circ T$.

If we let $(y(t)^T v(t))^T = T(x(t), u(t))$, then $y(t)$ and $v(t)$ satisfy Σ_0. Definition 2 indicates that we want to find a diffeomorphism T such that the diagram in Figure 1 commutes. Once we find such a diffeomorphism, we can apply linear system theory instead of nonlinear system theory.

Definition 3: Σ is said to be approximately linearizable with order ρ if there exist an open neighborhood $U(\subset \mathbb{R}^{N+1})$ of (x_e, u_e) and a diffeomorphism $T: U \to T(U)$ such that (i) $\bar{T} = (T_1, T_2, \ldots, T_N)$ are functions of $x_1, x_2, \ldots, x_N$ only, (ii) $T(x_e, u_e) = 0_{(N+1)\times 1}$, and (iii) $\bar{T} \circ f = g \circ T + O(x - x_e, u - u_e)^{\rho+1}$.

Thus, in Definition 3 we consider the following nearly linear discrete-time system:

$$\Sigma_0': \; y(t+1) = Ay(t) + bv(t) + O(x - x_e, u - u_e)^{\rho+1}$$

where the $N \times N$ matrix A and $N \times 1$ matrix b are the same as Σ_0. Clearly, local linearizability at (x_e, u_e) implies approximate linearizability with arbitrary order.

In Section II, some background material is reviewed and notations are defined. In Section III, necessary and sufficient conditions for approximate linearizability will be given for the system (1). Also, we will give a sufficient condition for local linearizability. We can define local linearizability and approximate linearizability for multi-input discrete-time systems similarly to Definitions 2 and 3. Then the multi-input case will be discussed in Section IV.

$$
\begin{tikzcd}
\mathbb{R}^N \arrow[d, "\overline{T}"'] & \mathbb{R}^{N+1} \arrow[l, "f"'] \arrow[d, "T"] \\
\mathbb{R}^N & \mathbb{R}^{N+1} \arrow[l, "g"]
\end{tikzcd}
$$

Figure 1

II. Preliminaries

In this section, notations and definitions to be used later will be mentioned. The Kronecker product is very useful in the field of matrix calculus (Graham 1981). First, define the Kronecker product $\otimes$ by

$$\underset{p\times q}{A} \otimes \underset{m\times n}{B} = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1q}B \\ a_{21}B & a_{22}B & \cdots & a_{2q}B \\ \vdots & \vdots & & \vdots \\ a_{p1}B & a_{p2}B & \cdots & a_{pq}B \end{bmatrix}_{(pm)\times(qn)},$$

where a_{ij} is the (i,j)-component of the $p\times q$ matrix A.

Define the derivative of a matrix with respect to a matrix by

$$D_A B = \begin{bmatrix} \frac{\partial}{\partial a_{11}}B & \frac{\partial}{\partial a_{12}}B & \cdots & \frac{\partial}{\partial a_{1q}}B \\ \frac{\partial}{\partial a_{21}}B & \frac{\partial}{\partial a_{22}}B & \cdots & \frac{\partial}{\partial a_{2q}}B \\ \vdots & \vdots & & \vdots \\ \frac{\partial}{\partial a_{p1}}B & \frac{\partial}{\partial a_{p2}}B & \cdots & \frac{\partial}{\partial a_{pq}}B \end{bmatrix}_{(mp)\times(nq)}$$

We also define

$$D_A^0 B = B, \quad D_A^1 B = D_A B, \text{ and}$$

$$D_A^{i+1} B = D_A(D_A^i B) \text{ for } i \geq 1.$$

Let $h(x)$ be a scalar real valued function of $x \in \mathbb{R}^N$. Then $(D_x^k h)(x)$ and $(D_{x^T}^k h)(x)$ are $N^k \times 1$ and $1\times N^k$ vectors, respectively.

<u>Fact</u> (Vetter 1970,1971): Using the definition of Kronecker product and derivative operations on matrices, Taylor's formula can be expressed by

$$h(x) = h(0) + \sum_{k=1}^{\ell} \frac{1}{k!} (D_{x^T}^k h(x))_{x=0} (x \otimes x \otimes \cdots \otimes x) + R_{\ell+1}(x^*),$$

where $R_{\ell+1}(x^*)$ is a remainder term.

Now define the $N^k \times N^k$ permutation matrix $U_{i_1,i_2,\ldots,i_k}$ as follows: the $(a_{i_1}-1)N^{k-1}+(a_{i_2}-1)N^{k-2}+\ldots+(a_{i_{k-1}}-1)N+a_{i_k}$-th column of $U_{i_1,i_2,\ldots,i_k}$ is the $(a_1-1)N^{k-1}+(a_2-1)N^{k-2}+\ldots+(a_{k-1}-1)N+a_k$-th column of $N^k \times N^k$ identity matrix $(I_{N^k \times N^k})$, for $1 \leq a_1,a_2,\ldots,a_k \leq N$ (the $\{a_i\}$ are related to the "base N" representation of the column). Here $\{i_1,i_2,\ldots,i_k\}$ is a permutation of $\{1,2,\ldots,k\}$. For example, when N=2 and k=3,

$$U_{123} = I_{8\times 8}$$

and

$$U_{321} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let A be a $p \times N^k$ matrix. Define the operator $\underset{k!}{\oplus}$ by

$$\underset{k!}{\oplus} A = A\Big(\sum_{\substack{\text{all permutations} \\ \{i_1,i_2,\ldots,i_k\} \text{ of } \{1,2,\ldots,k\}}} U_{i_1,i_2,\ldots,i_k}\Big)$$

For example, when A is a $p \times N^3$ matrix,

$$\underset{3!}{\oplus} A = A(U_{123}+U_{132}+U_{213}+U_{231}+U_{312}+U_{321}).$$

Let

$$\left\{\left(\frac{\partial f}{\partial u}\right)_{(0,0)}, \left(\frac{\partial f}{\partial x}\right)_{(0,0)}\left(\frac{\partial f}{\partial u}\right)_{(0,0)}, \ldots, \left(\frac{\partial f}{\partial x}\right)^{N-1}_{(0,0)}\left(\frac{\partial f}{\partial u}\right)_{(0,0)}\right\}$$

be linearly independent; that is, they form a basis for $\mathbb{R}^N$. Define $\zeta: \mathbb{R}^N \to \mathbb{R}$ by $\zeta(v) = \alpha_N$, where v is a $1 \times N$ row vector and

$$v^T = \sum_{i=1}^{N} \alpha_i \left(\frac{\partial f}{\partial x}\right)^{i-1}_{(0,0)}\left(\frac{\partial f}{\partial u}\right)_{(0,0)}.$$

That is, $\zeta(v)$ is the last coefficient of v^T with respect to the basis $\{w_1, w_2, \ldots, w_N\}$, where $w_i = (\frac{\partial f}{\partial x})^{i-1}_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}$, $1 \leq i \leq N$. Also define $\tilde{\zeta}: \mathbb{R}^{p \times N} \to \mathbb{R}^p$ by

$$\tilde{\zeta}(V) = \begin{bmatrix} \zeta(v_1) \\ \zeta(v_2) \\ \vdots \\ \zeta(v_p) \end{bmatrix}$$

where v_i is the i^{th} row of V.

III. Single-Input Case

In this section, our main results will be given. If $f(x,u)$ has an equilibrium point, without loss of generality, we can assume that $f(0,0)=0$; for, if not, let $\tilde{x}=x-x_e$ and $\tilde{u}=u-u_e$. Then $\tilde{x}(t+1)=\tilde{f}(\tilde{x}(t),\tilde{u}(t)) \triangleq f(\tilde{x}+x_e,\tilde{u}+u_e)-x_e$ with $\tilde{f}(0,0)=0$.

Let

$$\hat{f}^1(x,u)=f(x,u),$$

$$\hat{f}^{i+1}(x,u) = f(\hat{f}^i(x,u),0), \text{ for } 1 \leq i \leq N-1.$$

$\hat{f}^i(x,u)$ represents the effect of an input u at $t=0$ on the state at $t=i$. $\hat{f}^i(x,u)$ is essential for solving many problems arising in discrete time nonlinear systems.

Lemma 1: Σ is locally linearizable at $(0,0)$ if and only if there exists a C^∞ function $h:W(\subset \mathbb{R}^N) \to \mathbb{R}$ such that (i) W is an open neighborhood of $0\in\mathbb{R}^N$, (ii) $D_u(h\circ\hat{f}^i) \equiv 0$ on some neighborhood of $0\in\mathbb{R}^{N+1}$, for $1 \leq i \leq N-1$,

$$\text{(iii)} \qquad \det \begin{bmatrix} (\frac{\partial h}{\partial x})_{x=0} \\ (\frac{\partial(h\circ\hat{f})}{\partial x})_{(0,0)} \\ \vdots \\ (\frac{\partial(h\circ\hat{f}^{N-1})}{\partial x})_{(0,0)} \end{bmatrix} \neq 0,$$

(iv) $(D_u(h\circ\hat{f}^N))_{(0,0)} \neq 0$, and (v) $h(0)=0$.

Proof: Necessity: Suppose that Σ is locally linearizable. Thus we have a diffeomorphism T. Let $h(x)=T_1(x)$. (Since $T_1(x,u)$ depends only on x, we can write $T_1(x)$ instead of $T_1(x,u)$.) Note that $T_2=T_1\circ f$. Since $D_u(T_2)\equiv 0$ on some neighborhood of the origin, $D_u(T_1\circ f)\equiv 0$ on some neighborhood of the origin. From now on, for convenience, we will omit "on some neighborhood of the origin". Note that $T_3=T_2\circ f=T_1\circ\hat{f}^2$. (Actually, we can write $T_3=T_1\circ f^2$, because $T_1\circ f$ depends only on x. But $\hat{f}^2$ is used, for consistency of notation.) Since $D_u(T_3)\equiv 0$, $D_u(T_1\circ\hat{f}^2)\equiv 0$. Proceeding in this manner, since $T_N=T_{N-1}\circ f=\ldots=T_1\circ\hat{f}^{N-1}$ and $D_u(T_N)\equiv 0$,

$D_u(T_1 \circ \hat{f}^{N-1}) \equiv 0$. Thus we have shown that $D_u(T_1 \circ \hat{f}^i) \equiv 0$, for $1 \le i \le N-1$. Since T is a diffeomorphism, $T_i = T_1 \circ \hat{f}^{i-1}$ for $2 \le i \le N+1$, and $T_1, T_2, \ldots, T_N$ depend only on x,

$$\det \begin{bmatrix} (\frac{\partial h}{\partial x})_{x=0} \\ (\frac{\partial (h \circ \hat{f})}{\partial x})_{(0,0)} \\ \vdots \\ (\frac{\partial (h \circ \hat{f}^{N-1}}{\partial x})_{(0,0)} \end{bmatrix} \ne 0$$

and $D_u(h \circ \hat{f}^N) \ne 0$. Since $T_1(0,0) = 0$, $h(0) = 0$.

<u>Sufficiency</u>: Suppose that there exists $h: \mathbb{R}^N \to \mathbb{R}$ satisfying the given conditions. Let $T_i(x) = h \circ \hat{f}^{i-1}$, for $1 \le i \le N+1$. Then it can be easily checked that $\bar{T} \circ f = g \circ T$ and $T(0,0) = 0$. Since $\det((\frac{\partial T}{\partial (x,u)})_{(0,0)}) \ne 0$, there exists an open neighborhood U of (0,0) such that $T: U \to T(U)$ is a diffeomorphism by the inverse function theorem. (Q.E.D.)

Let $\xi = \binom{x}{u}$.

<u>Lemma 2</u>: Σ is <u>approximately linearizable</u> with order ρ if and only if there exists a C^∞ function $h: W(\subset \mathbb{R}^N) \to \mathbb{R}$ such that (i) W is an open neighborhood of $0 \in \mathbb{R}^N$, (ii) $(D_\xi^j D_u(h \circ \hat{f}^i))_{(0,0)} = 0_{(N+1)^j \times 1}$ for $1 \le i \le N-1$ and $0 \le j \le \rho - 1$,

(iii)
$$\det \begin{bmatrix} (\frac{\partial h}{\partial x})_{x=0} \\ (\frac{\partial (h \circ \hat{f})}{\partial x})_{(0,0)} \\ \vdots \\ (\frac{\partial (h \circ \hat{f}^{N-1})}{\partial x})_{(0,0)} \end{bmatrix} \ne 0,$$

(iv) $(D_u(h \circ \hat{f}^N))_{(0,0)} \ne 0$, and (v) $h(0) = 0$.

<u>Proof</u>: <u>Necessity</u>: Suppose Σ is approximately linearizable with order ρ.

Let $h(x) = T_1(x)$. By definition, $T_1 \circ f(x,u) = T_2(x) + O(x,u)^{\rho+1}$. So $(D_\xi^j(D_u(h \circ f))_{(0,0)} = 0$ for $0 \leq j \leq \rho-1$. Note that $T_1 \circ \hat{f}^2 = T_2 \circ f + O(x,u)^{\rho+1}$. Since $T_2 \circ f(x,u) = T_3(x) + O(x,u)^{\rho+1}$ by definition, $T_1 \circ \hat{f}^2 = T_3(x) + O(x,u)^{\rho+1}$. Thus $(D_\xi^j D_u(h \circ \hat{f}^2))_{(0,0)} = 0$ for $0 \leq j \leq \rho-1$. Proceeding in this manner, we can show that $(D_\xi^j D_u(h \circ \hat{f}^i))_{(0,0)} = 0$ for $1 \leq i \leq N-1$ and $0 \leq j \leq \rho-1$. Note that $\left(\frac{\partial T_i}{\partial x} \right)_{(0,0)} = \left(\frac{\partial}{\partial x} (h \circ \hat{f}^{i-1}) \right)_{(0,0)}$ for $1 \leq i \leq N$. Since

$$\det \begin{bmatrix} \left(\frac{\partial T_1}{\partial x} \right)_{(0,0)} \\ \left(\frac{\partial T_2}{\partial x} \right)_{(0,0)} \\ \vdots \\ \left(\frac{\partial T_N}{\partial x} \right)_{(0,0)} \end{bmatrix} \neq 0, \qquad \det \begin{bmatrix} \left(\frac{\partial h}{\partial x} \right)_{x=0} \\ \left(\frac{\partial (h \circ f)}{\partial x} \right)_{(0,0)} \\ \vdots \\ \left(\frac{\partial (h \circ \hat{f}^{N-1})}{\partial x} \right)_{(0,0)} \end{bmatrix} \neq 0.$$

It can be easily shown that $T_1 \circ \hat{f}^N(x,u) = T_{N+1}(x,u) + O(x,u)^{\rho+1}$. Thus $(D_u(h \circ \hat{f}^N))_{(0,0)} = (D_u T_{N+1}(x,u))_{(0,0)} \neq 0$. *Finally,* $h(0) = T_1(0) = 0$.

<u>Sufficiency</u>: (by construction) Let

$$T_1(x) = h(x)$$

$$T_2(x) = \sum_{k=1}^{\rho} \frac{1}{k!} (D_x^k T(h \circ \hat{f}))_{(0,0)} \, (\overbrace{x \otimes x \otimes \ldots \otimes x}^{k \text{ times}})$$

$$T_3(x) = \sum_{k=1}^{\rho} \frac{1}{k!} (D_x^k T(h \circ \hat{f}^2))_{(0,0)} \, (x \otimes x \otimes \ldots \otimes x)$$

$$\vdots$$

$$T_N(x) = \sum_{k=1}^{\rho} \frac{1}{k!} (D_x^k T(h \circ \hat{f}^{N-1}))_{(0,0)} \, (x \otimes x \otimes \ldots \otimes x)$$

$$T_{N+1}(x,u) = T_1 \circ \hat{f}^N(x,u).$$

Then it can be easily checked that T as defined above satisfies the conditions of Definition 3. Q.E.D.

Now note that

$$(D_\xi^m D_u(h\circ\hat{f}^i))_{(0,0)} = \sum_{\ell=0}^{m} (B_{m,\ell}^i)_{(0,0)}(D_x^{\ell+1} h)_{x=0}, \tag{2}$$

where

$$B_{m,0}^i = D_\xi^m (D_u \hat{f}^i)^T$$

and

$$B_{m,\ell}^i = \sum_{k_1=1}^{m-\ell+1} \sum_{k_2=1}^{k_1} \cdots \sum_{k_\ell=1}^{k_{\ell-1}} D_\xi^{m-\ell+1-k_1}(D_\xi \hat{f}^{i^T} \otimes D_\xi^{k_1-k_2}(D_\xi \hat{f}^{i^T} \otimes D_\xi^{k_2-k_3}$$

$$\cdot(D_\xi \hat{f}^{i^T} \otimes \ldots \otimes D_\xi^{k_{\ell-1}-k_\ell}(D_\xi \hat{f}^{i^T} \otimes D_\xi^{k_\ell-1} D_u \hat{f}^{i^T}) \ldots)), \text{ for } 1 \leq \ell \leq m.$$

(For a proof of (2), see the Appendix.) Let

$$A_k = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{bmatrix}$$

where A_{ij} is an $(N-1)(N+1)^i \times N^{j+1}$ submatrix defined by

$$A_{ij} = 0_{(N-1)(N+1)^i \times N^{j+1}}, \quad \text{if } i < j,$$

$$A_{ij} = \underset{(j+1)!}{\bullet} \begin{bmatrix} (B_{ij}^1)_{(0,0)} \\ (B_{ij}^2)_{(0,0)} \\ \vdots \\ (B_{ij}^{N-1})_{(0,0)} \end{bmatrix}, \quad \text{if } i \geq j.$$

Let the $(N-1)(N+1)^i \times 1$ vector β^i be defined by

$$\beta^{i} = \tilde{\zeta}\left(\begin{bmatrix}(B^{1}_{i,0})(0,0)\\ (B^{2}_{i,0})(0,0)\\ \vdots\\ (B^{N-1}_{i,0})(0,0)\end{bmatrix}\right)$$

(See Section II for the definitions of $\underset{j!}{\circledast}$ and $\tilde{\zeta}$.) Also, let $\beta_k = (\beta^{1^T}\ \beta^{2^T}\ \dots\ \beta^{k^T})^T$.

With these preliminaries, we can state our main theorems.

Theorem 3: Σ is approximately linearizable with order $\rho(\geq 2)$ if and only if

(i) $\left\{(\frac{\partial f}{\partial u})_{(0,0)}, (\frac{\partial f}{\partial x})_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}, \dots, (\frac{\partial f}{\partial x})^{N-1}_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}\right\}$ are linearly independent, and

(ii) $\beta_{\rho-1} \in \mathrm{Image}(A_{\rho-1})$.

Proof:

Necessity: Suppose that Σ is approximately linearizable with order ρ. Then there exists a function $h(x)$ satisfying (i)-(v) of Lemma 2; in particular, $(D_u(h\circ\hat{f}^{i}))_{(0,0)} = (\frac{\partial h}{\partial x})_{x=0}\ (\frac{\partial f}{\partial x})^{i-1}_{(0,0)}\ (\frac{\partial f}{\partial u})_{(0,0)} = 0$ for $1 \leq i \leq N-1$, and $(\frac{\partial h}{\partial x})_{x=0}\ (\frac{\partial f}{\partial x})^{N-1}_{(0,0)}\ (\frac{\partial f}{\partial u})_{(0,0)} \neq 0$.

Assume that $\left\{(\frac{\partial f}{\partial u})_{(0,0)},(\frac{\partial f}{\partial x})_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}, \dots, (\frac{\partial f}{\partial x})^{N-1}_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}\right\}$ are not linearly independent. Then there exists k such that $1 \leq k \leq N-1$ and $(\frac{\partial f}{\partial x})^{k}_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)} = \sum_{j=0}^{k-1} \alpha_j (\frac{\partial f}{\partial x})^{j}_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}$ for some constants $\{\alpha_j\}^{k-1}_{j=0}$. Thus $(\frac{\partial f}{\partial x})^{N-1}(\frac{\partial f}{\partial u})_{(0,0)} = \sum_{j=0}^{k-1} \alpha_j (\frac{\partial f}{\partial x})^{N-1-k+j}_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}$, and

$(\frac{\partial h}{\partial x})_{x=0}(\frac{\partial f}{\partial x})^{N-1}_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)} = 0$. This is a contradiction, which implies that $\{(\frac{\partial f}{\partial u})_{(0,0)}, (\frac{\partial f}{\partial x})_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}, \ldots, (\frac{\partial f}{\partial x})^{N-1}_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}\}$ are linearly independent.

Recall that $(D_u(h\circ\hat{f}^i))_{(0,0)} = (D^m_\xi D_u(h\circ\hat{f}^i))_{(0,0)} = 0$ for m=0 and $1 \le i \le N-1$. Since $\{(\frac{\partial f}{\partial u})_{(0,0)}, (\frac{\partial f}{\partial x})_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}, \ldots, (\frac{\partial f}{\partial x})^{N-1}_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}\}$ are linearly independent, $(D_x h)_{x=0}$ is uniquely determined up to a constant multiple (i.e., $(D_x h)_{x=0} = \alpha c$, where the scalar $\alpha(\neq 0)$ is arbitrary and the $N\times 1$ column vector c satisfies $c^T(\frac{\partial f}{\partial x})^i_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)} = 0$, for $0 \le i \le N-2$, and $c^T(\frac{\partial f}{\partial x})^{N-1}_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)} = 1$).

Now by (ii) and (iv) of Lemma 2, $(D^m_\xi D_u(h\circ\hat{f}^i))_{(0,0)} = 0$ for $1 \le m \le \rho-1$ and $1 \le i \le N-1$. From (2), we obtain

$$\begin{bmatrix} B^1_{1,1} & & & \\ B^2_{1,1} & 0 & \cdots & 0 \\ \vdots & & & \\ B^{N-1}_{1,1} & & & \\ B^1_{2,1} & B^1_{2,2} & & \\ \vdots & \vdots & 0 \;\cdots & 0 \\ B^{N-1}_{2,1} & B^{N-1}_{2,2} & & \\ \vdots & \vdots & & 0 \\ B^1_{\rho-1,1} & B^1_{\rho-1,2} & & B^1_{\rho-1,\rho-1} \\ \vdots & \vdots & \cdots & \vdots \\ B^{N-1}_{\rho-1,1} & B^{N-1}_{\rho-1,2} & & B^{N-1}_{\rho-1,\rho-1} \end{bmatrix}_{(0,0)} \begin{bmatrix} (D^2_x h)_{x=0} \\ (D^3_x h)_{x=0} \\ \vdots \\ (D^\rho_x h)_{x=0} \end{bmatrix} = - \begin{bmatrix} B^1_{1,0} \\ B^2_{1,0} \\ \vdots \\ B^{N-1}_{1,0} \\ B^1_{2,0} \\ \vdots \\ B^{N-1}_{2,0} \\ \vdots \\ B^1_{\rho-1,0} \\ \vdots \\ B^{N-1}_{\rho-1,0} \end{bmatrix}_{(0,0)} (D^1_x h)_{x=0} \qquad (3)$$

Since $(B^1_{1,0})_{(0,0)}(D^1_x h)_{x=0} = (D^1_\xi D_u \hat{f}^T)_{(0,0)}(D^1_x h)_{x=0} = \alpha\ ((D^1_\xi D_u \hat{f}^T)_{(0,0)})$,

$$\begin{bmatrix} B^1_{1,0} \\ B^2_{1,0} \\ \vdots \\ B^{N-1}_{1,0} \end{bmatrix}_{(0,0)} (D^1_x h)_{x=0} = \alpha\beta^1$$

Thus the right-hand side of equation (3) is $-\alpha[(\beta^1)^T,\ldots,(\beta^{\rho-1})^T]^T = -\alpha\beta^T_{\rho-1}$. It follows that $\beta_{\rho-1}$ is in the image of the matrix on the left-hand side of (3). However, the $\{D_x^k h\}$ are constrained because, for example, $\partial^2 h/\partial x_i^2 \partial x_j = \partial^2 h/\partial x_j \partial x_i$. Hence, the stronger condition $\beta_{\rho-1} \in \text{Image}\ (A_{\rho-1})$ holds, as is proved in the Appendix in Lemma A.2.

Sufficiency: Suppose that (i) and (ii) above are true. By (i), there exists an $N\times 1$ vector C_1 such that $C_1^T(\frac{\partial f}{\partial x})^i_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)} = 0$ for $0 \le i \le N-2$ and $C_1^T(\frac{\partial f}{\partial x})^{N-1}_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)} = 1$. By (ii), there exist $C_2, C_3, \ldots, C_\rho$ such that

$$A_{\rho-1} \begin{bmatrix} \frac{1}{2!} C_2 \\ \frac{1}{3!} C_3 \\ \vdots \\ \frac{1}{\rho!} C_\rho \end{bmatrix} = -\beta_{\rho-1}$$

where C_i is an $N^i \times 1$ vector. Let

$$h(x) = \sum_{i=1}^{\rho} \frac{1}{i!} C_i^T (\underbrace{x \otimes x \otimes \ldots \otimes x}_{i \text{ times}}).$$

Then it can be easily checked that $(D^j_\xi D_u(h\circ\hat{f}^i))_{(0,0)} = 0$, for $1 \le i \le N-1$ and $0 \le j \le \rho-1$. Clearly $(D_u(h\circ\hat{f}^N))_{(0,0)} = (D^1_x h)^T_{x=0}(\frac{\partial f}{\partial x})^{N-1}_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)} = 1 \neq 0$.

Now assume that

$$\det \begin{bmatrix} (\frac{\partial h}{\partial x})_{x=0} \\ (\frac{\partial(h\circ\hat{f})}{\partial x})_{(0,0)} \\ \vdots \\ (\frac{\partial(h\circ\hat{f}^{N-1})}{\partial x})_{(0,0)} \end{bmatrix} = 0.$$

Then there exists k such that $1 \le k \le N-1$ and $(D_x^1 h)^T_{x=0}(\frac{\partial f}{\partial x})^k_{(0,0)} = \sum_{i=0}^{k-1}$ $\cdot\alpha_i(D_x^1 h)^T_{x=0}(\frac{\partial f}{\partial x})^i_{(0,0)}$, for some $\{\alpha_i\}_{i=0}^{k-1}$. Thus $(D_x^1 h)^T_{x=0}(\frac{\partial f}{\partial x})^{N-1}_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}$ $= \sum_{i=0}^{k-1} \alpha_i (D_x^1 h)^T_{x=0}(\frac{\partial f}{\partial x})^{N-1-k+i}_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)} = 0$. This is a contraction, which implies (iii) of Lemma 2. Hence, by Lemma 2, Σ is approximately linearizable with order ρ. (Q.E.D.)

Remark: Σ is approximately linearizable with order 1 if and only if (i) of Theorem 3 holds, just as in the continuous case (Krener 1984).

Now a sufficient condition for local linearizability is given in the following theorem.

Theorem 4: Suppose that f(x,u) of Σ is an analytic $\mathbb{R}^N$-valued function. Σ is locally linearizable at (0,0) if

(i) $\{(\frac{\partial f}{\partial u})_{(0,0)}, (\frac{\partial f}{\partial x})_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}, \ldots, (\frac{\partial f}{\partial x})^{N-1}_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}\}$ are linearly independent, and

(ii) there exists $k(<\infty)$ such that $\beta_\ell \in \operatorname{span}(C_\ell^k)$ for all $\ell \ge 1$, where C_ℓ^k is composed of the first k columns of A_ℓ.

Proof: By (i), there exists an N×1 vector c_1 such that $c_1^T(\frac{\partial f}{\partial x})^i_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}$ $= 0$ for $0 \le i \le N-2$ and $c_1^T(\frac{\partial f}{\partial x})^{N-1}_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)} = 1$. By (ii), there exist $c_2, c_3, \ldots, c_j$ such that $j < \infty$ and

$$A_\ell \begin{bmatrix} \frac{1}{2!} c_2 \\ \frac{1}{3!} c_3 \\ \vdots \\ \frac{1}{j!} c_j \\ 0 \\ \vdots \\ 0 \end{bmatrix} = -\beta_\ell, \text{ for } \ell \geq j,$$

where c_i is an $N^i \times 1$ vector. Let

$$h(x) = \sum_{i=1}^{j} \frac{1}{i!} c_i^T \overbrace{(x \otimes x \otimes \ldots \otimes x)}^{i \text{ terms}}.$$

Then it can be easily checked that $(D_\xi^s D_u(h \circ \hat{f}^i))_{(0,0)} = 0$ for $1 \leq i \leq N-1$ and $s \geq 0$. Since both $h(x)$ and $\hat{f}^i$ are analytic, $h \circ \hat{f}^i(x,u)$ are analytic, for $1 \leq i \leq N-1$. Thus $D_u(h \circ \hat{f}^i) \equiv 0$, for $1 \leq i \leq N-1$. As in the sufficiency proof in Theorem 3, it can be shown that h satisfies the other conditions of Lemma 1. (Q.E.D.)

It is easy to see that (ii) of Theorem 4 implies (ii) of Theorem 3.

Example 1: Consider the following discrete-time nonlinear system

$$\Sigma: \quad \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} x_2(t)+2x_1(t)u(t)+u(t)^2 \\ x_1(t)+u(t) \end{bmatrix} = f(x(t),u(t))$$

Since $(\frac{\partial f}{\partial u})_{(0,0)} = \binom{0}{1}$ and $(\frac{\partial f}{\partial x})_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)} = \binom{1}{0}$, (i) of Theorem 4 is satisfied. Note that

$$\beta^1 = ((D_\xi D_u f^T)_{(0,0)}) = \tilde{\zeta}\left(\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 2 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

and $\beta^i = 0_{3^i \times 1}$ for $i \geq 2$.

$$(B^1_{11})_{(0,0)} = (D_\xi f^T)_{(0,0)} \otimes (D_u f^T)_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes (0\ 1) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_1 = \frac{\circledast}{2!} (B^1_{11})_{(0,0)} = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

It can be easily checked that all elements of the 4-th column of A_{i1} are 0 for $i \geq 2$. Thus $\beta_\ell \in \text{span}(C^4_\ell)$ for $\ell \geq 1$, where C^4_ℓ is composed of the first 4 columns of A_ℓ. Therefore (ii) of Theorem 4 is also satisfied. Hence Σ is locally linearizable at (0,0). Actually we can construct a diffeomorphism $T = (T_1, T_2, T_3)$ in the way that is given in the proof of Theorem 4. Since

$$A_\ell \begin{bmatrix} \frac{1}{2!} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix} \\ \frac{1}{3!}\, 0_{8\times 1} \\ \vdots \\ \frac{1}{(\ell+1)!}\, 0_{2^{\ell+1}\times 1} \end{bmatrix} = -\beta_\ell \quad \text{for } \ell \geq 1,$$

$c_2^T = (0\ 0\ 0\ -2)$. Clearly $c_1^T = (1\ 0)$. Thus

$$T_1(x) = x_1 - \frac{2}{2!} x_2^2 = x_1 - x_2^2$$

$$T_2(x) = T_1 \circ f(x,u) = x_2 - x_1^2$$

$$T_3(x,u) = T_1 \circ \hat{f}^2(x,u) = x_1 + u - (x_2 + 2x_1 u + u^2)^2.$$

<u>Example 2</u>: Consider

$$\Sigma: \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} x_2(t) + 2x_1(t)u(t) + x_1(t)^2 u(t) + u(t)^2 \\ x_1(t) + u(t) \end{bmatrix} = f(x(t), u(t))$$

Clearly (i) of Theorem 3 is satisfied, because $(\frac{\partial f}{\partial u})_{(0,0)}$ and $(\frac{\partial f}{\partial x})_{(0,0)}(\frac{\partial f}{\partial u})_{(0,0)}$ are the same as in Example 1. Since $(D_\xi^1 D_u f^T)_{(0,0)} =$

$\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 2 & 0 \end{bmatrix}$, $\beta^1 = \tilde{\zeta}((D_\xi D_u f^T)_{(0,0)}) = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$. Since $(D_\xi^2 D_u f^T)_{(0,0)} =$

$\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\beta^2 = (2\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)^T$.

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$(B_{22}^1)_{(0,0)} = (D_\xi f^T)_{(0,0)} \otimes (D_\xi f^T)_{(0,0)} \otimes (D_u f^T)_{(0,0)}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes (0 \quad 1)$$

$$= \begin{bmatrix} 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&1&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix} \quad \therefore A_{22} = \underset{3!}{\bullet} (B_{22}^1)_{(0,0)}$$

$$= \begin{bmatrix} 0&0&0&0&0&0&0&6 \\ 0&0&0&2&0&2&2&0 \\ 0&0&0&0&0&0&0&6 \\ 0&0&0&2&0&2&2&0 \\ 0&2&2&0&2&0&0&0 \\ 0&0&0&2&0&2&2&0 \\ 0&0&0&0&0&0&0&6 \\ 0&0&0&2&0&2&2&0 \\ 0&0&0&0&0&0&0&6 \end{bmatrix}$$

$$(B^1_{21})_{(0,0)} = (D_\xi(D_\xi f^T \otimes D_u f^T))_{(0,0)} + (D_\xi f^T \otimes D_\xi D_u f^T)_{(0,0)}$$

$$= (D^2_\xi f^T \otimes D_u f^T)_{(0,0)} + \begin{bmatrix} D_\xi f^T \otimes D_{x_1} D_u f^T \\ D_\xi f^T \otimes D_{x_2} D_u f^T \\ D_\xi f^T \otimes D_u D_u f^T \end{bmatrix} + (D_\xi f^T \otimes D_\xi D_u f^T)_{(0,0)}$$

$$= \begin{bmatrix} 0 & 0 & 4 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 \end{bmatrix}$$

$$A_{21} = \frac{\bullet}{2!} (B^1_{21})_{(0,0)} = \begin{bmatrix} 0 & 4 & 4 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 6 & 6 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 6 & 6 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 6 & 6 & 0 \end{bmatrix}$$

Since $\beta_1 \in$ Image (A_1), Σ is approximately linearizable with $\rho=2$. However, since $\beta_2 \notin$ Image (A_2), it is not approximately linearizable with $\rho=3$. Thus it is also not locally linearizable. Let

$$T_1 = x_1 - x_2^2$$

$$T_2 = x_2 - x_1^2$$

$$T_3 = x_1 + u - (x_2 + 2x_1 u + x_1^2 u + u^2)^2$$

Then

$$
\begin{aligned}
y_1(t+1) \triangleq T_1(x(t+1)) &= x_2(t) + 2x_1(t)u(t) + x_1(t)^2 u(t) + u(t)^2 - (x_1(t) + u(t))^2 \\
&= x_2(t) - x_1(t)^2 + x_1(t)^2 u(t) \\
&= T_2(x(t)) + 0(x,u)^3 \\
&= y_2(t) + 0(x,u)^3 \\
y_2(t+1) \triangleq T_2(x(t+1)) &= T_3(x(t),u(t)) \triangleq v(t)
\end{aligned}
$$

IV. Multi-Input Case

The results in Section III can be easily generalized to the multi-input case. Thus, in this section we give (without proof) a sufficient condition for local linearizability and a necessary and sufficient condition for approximate linearizability by state feedback and coordinate change for a multi-input nonlinear discrete-time system (for proof, see Lee 1986).

Consider a multi-input nonlinear discrete-time system of the form

$$\Sigma: \quad x(t+1) = f(x(t),u(t)) \tag{4}$$

where $x(t) \in \mathbb{R}^N$, $u(t) \in \mathbb{R}^m$, and $f(x,u)$: $\mathbb{R}^{N+m} \to \mathbb{R}^N$ is a C^∞ $\mathbb{R}^N$-valued function. Also, consider the following multi-input linear discrete-time system Σ_0:

$$\Sigma_0: \; y(t+1) = Ay(t) + Bv(t) = g(y(t),v(t)),$$

where $y(t) \in \mathbb{R}^N$, $v(t) \in \mathbb{R}^m$, $A = \text{block diag } \{A_{11}, A_{22}, \ldots, A_{mm}\}$,

$$A_{ii} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad (K_i \times K_i \text{ matrix}),$$

$$\sum_{i=1}^{m} K_i = N,$$

$B = \text{block diag } \{b_1, b_2, \ldots, b_m\}$, and

$$b_i = (0 \; \ldots \; 0 \;\; 1)^T. \qquad (K_i \times 1 \text{ matrix})$$

<u>Definition 4</u>: Σ is said to be locally linearizable at (x_e,u_e) if there exist indices $\{K_i\}_{i=1}^m$, an open neighborhood $U(\subset \mathbb{R}^{N+m})$ of an equilibrium

point (x_e, u_e) and a diffeomorphism $T: U \to T(U)$ such that

(i) $\bar{T} = (T_1, T_2, \ldots, T_N)$ are functions of $x_1, x_2, \ldots, x_N$ only,

(ii) $T(x_e, u_e) = 0_{(N+m)\times 1}$, and

(iii) $\bar{T} \circ f = g \circ T$.

If we let $\binom{y(t)}{v(t)} = T(x(t), u(t))$, then $y(t)$ and $v(t)$ satisfy the relation Σ_0.

Definition 5: Σ is said to be approximately linearizable with order ρ if there exist indices $\{K_i\}_{i=1}^m$, an open neighborhood U $(\subset \mathbb{R}^{N+m})$ of an equilibrium point (x_e, u_e) and a diffeomorphism $T: U \to T(U)$ such that

(i) $\bar{T} = (T_1, T_2, \ldots, T_N)$ are functions of $x_1, x_2, \ldots, x_N$ only,

(ii) $T(x_e, u_e) = 0_{(N+m)\times 1}$, and

(iii) $\bar{T} \circ f = g \circ T + 0(x - x_e, u - u_e)^{\rho+1}$.

Thus in Definition 5 we consider the following nearly linear multi-input discrete-time system:

$$\Sigma_0': y(t+1) = Ay(t) + Bv(t) + 0(x - x_e, u - u_e)^{\rho+1},$$

where the $N\times N$ matrix A and $N\times m$ matrix B are as in Σ_0.

Now we state the generalized version of Lemmas 1 and 2 and Theorems 3 and 4. Just as in the single-input case, we can assume $f(0,0) = 0$ without loss of generality, if f has an equilibrium point. Also, define $\hat{f}^i(x,u)$ in the same way as in the previous section.

Lemma 5: Σ is locally linearizable at $(0,0)$ if and only if there exist $\{K_i\}_{i=1}^m$ and C^∞ functions $h_1(x), h_2(x), \ldots, h_m(x)$: $W(\subset \mathbb{R}^N) \to \mathbb{R}$ such that

(i) W is an open neighborhood of $0 \in \mathbb{R}^N$,

(ii) $D_u(h_j \circ \hat{f}^i) \equiv 0$, for $1 \leq j \leq m$ and $1 \leq i \leq k_j - 1$,

(iii)

$$\det \begin{bmatrix} (\frac{\partial h_1}{\partial x})_{x=0} \\ (\frac{\partial h_1 \circ f}{\partial x})_{(0,0)} \\ (\frac{\partial h_1 \circ f^{K_1-1}}{\partial x})_{(0,0)} \\ \vdots \\ (\frac{\partial h_m}{\partial x})_{x=0} \\ \vdots \\ (\frac{\partial (h_m \circ f^{K_m-1})}{\partial x})_{(0,0)} \end{bmatrix} \neq 0$$

(iv)

$$\det \begin{bmatrix} (\frac{\partial (h_1 \circ f^{K_1})}{\partial u})_{(0,0)} \\ \vdots \\ (\frac{\partial (h_m \circ f^{K_m})}{\partial u})_{(0,0)} \end{bmatrix} \neq 0$$

and

(v) $h_j(0) = 0$ for $1 \leq j \leq m$.

Let $\xi = (x^T, u^T)^T$. Thus ξ is a $(N+m)\times 1$ vector.

<u>Lemma 6</u>: Σ is approximately linearizable with order ρ if and only if there exist $\{K_i\}_{i=1}^m$ and C^∞ functions $h_1(x), h_2(x), \ldots, h_m(x)$: $W(\subset \mathbb{R}^N) \to \mathbb{R}$ such that

(i) W is an open neighborhood of $0 \in \mathbb{R}^N$,

(ii) $(D_\xi^K D_u(h_j \circ \hat{f}^i))_{(0,0)} = 0$ for $1 \leq j \leq m$, $1 \leq i \leq k_j - 1$, and $0 \leq k \leq \rho-1$, and

(iii), (iv), and (v) of Lemma 5 are satisfied.

Let

$$E = \{(\tfrac{\partial f}{\partial u_1})(0,0),\ (\tfrac{\partial f}{\partial x})(0,0)(\tfrac{\partial f}{\partial u_1})(0,0),\dots,(\tfrac{\partial f}{\partial x})^{K_1-1}(0,0)(\tfrac{\partial f}{\partial u_1})(0,0),\ (\tfrac{\partial f}{\partial u_2})(0,0),$$

$$\dots,(\tfrac{\partial f}{\partial x})^{K_2-1}(0,0)(\tfrac{\partial f}{\partial u_2})(0,0),\dots,(\tfrac{\partial f}{\partial u_m})(0,0),\dots,(\tfrac{\partial f}{\partial x})^{K_m-1}(0,0)(\tfrac{\partial f}{\partial u_m})(0,0)\}$$

$$E_i = \{(\tfrac{\partial f}{\partial u_1})(0,0),\ (\tfrac{\partial f}{\partial x})(0,0)(\tfrac{\partial f}{\partial u_1})(0,0),\dots,(\tfrac{\partial f}{\partial x})^{K_i-2}(0,0)(\tfrac{\partial f}{\partial u_1})(0,0),\ (\tfrac{\partial f}{\partial u_2})(0,0),$$

$$\dots,(\tfrac{\partial f}{\partial x})^{K_i-2}(0,0)(\tfrac{\partial f}{\partial u_2})(0,0),\dots,(\tfrac{\partial f}{\partial u_m})(0,0),\dots,(\tfrac{\partial f}{\partial x})^{K_i-2}(0,0)(\tfrac{\partial f}{\partial u_m})(0,0)\},$$

$i=1,\dots,m$.

Suppose that the elements of E are linearly independent; that is, they form a basis for $\mathbb{R}^N$. Let $\sigma_i = \sum_{j=1}^{i} K_j$ for $1 \leq i \leq m$. Define $\zeta^i(v)$: $\mathbb{R}^N \to \mathbb{R}$ by $\zeta^i(v) = \alpha_{\sigma_i}$, where v is a 1×N row vector and

$$v^T = \alpha_1(\tfrac{\partial f}{\partial u_1})(0,0) + \dots + \alpha_{\sigma_1}(\tfrac{\partial f}{\partial x})^{K_1-1}(0,0)(\tfrac{\partial f}{\partial u_1})(0,0) + \alpha_{\sigma_1+1}(\tfrac{\partial f}{\partial u_2})(0,0) + \cdots$$
$$+ \alpha_{\sigma_2}(\tfrac{\partial f}{\partial x})^{K_2-1}(0,0)(\tfrac{\partial f}{\partial u_2})(0,0) + \dots + \alpha_{\sigma_{m-1}+1}(\tfrac{\partial f}{\partial u_m})(0,0) + \cdots$$
$$+ \alpha_{\sigma_m}(\tfrac{\partial f}{\partial x})^{K_m-1}(0,0)(\tfrac{\partial f}{\partial u_m})(0,0).$$

Also, define $\tilde{\zeta}^i$: $\mathbb{R}^{p\times N} \to \mathbb{R}^p$, for $i=1,2,\dots,m$, by

$$\tilde{\zeta}^i(V) = (\zeta^i(v_1)\ \zeta^i(v_2)\ \dots\ \zeta^i(v_p))^T$$

where v_j is the j^{th} row of V. Let

$$\beta_j^i = \tilde{\zeta}^i\left(\begin{bmatrix}(D_\xi^j D_u f^T)(0,0)\\ (D_\xi^j D_u \hat{f}^{2T})(0,0)\\ \vdots \\ (D_\xi^j D_u \hat{f}^{K_i-1^T})(0,0)\end{bmatrix}\right)$$

Also, let $\gamma_k^i = ((\beta_1^i)^T \; (\beta_2^i)^T \; \ldots \; (\beta_k^i)^T)^T$. Let

$$D_k^{\ell} = \begin{bmatrix} D_{11}^1 & D_{12}^1 & \cdots & D_{1k}^1 \\ D_{21}^1 & D_{22}^1 & \cdots & D_{2k}^1 \\ \vdots & \vdots & & \vdots \\ D_{k1}^1 & D_{k2}^1 & \cdots & D_{kk}^1 \end{bmatrix}$$

where $D_{ij}^{\ell} = 0_{m(k_{\ell}-1)(N+m)^i \times N^{j+1}}$ if $i < j$, and

$$D_{ij}^{\ell} = \frac{\bullet}{(j+1)!} \begin{bmatrix} (B_{ij}^1)(0,0) \\ (B_{ij}^2)(0,0) \\ \vdots \\ (B_{ij}^{k_{\ell}-1})(0,0) \end{bmatrix} \text{if } i \geq j$$

<u>Theorem 7</u>: Σ is approximately linearizable with order $\rho(\geq 2)$ if and only if there exist $\{K_i\}_{i=1}^m$ such that

(i) the elements of E are linearly independent,

(ii) span E_i = span $(E_i \cap E)$ for $1 \leq i \leq m$, and

(iii) $\gamma_{\rho-1}^{\ell} \in$ Image $(D_{\rho-1}^{\ell})$ for $1 \leq \ell \leq m$.

<u>Remark</u>: Σ is approximately linearizable with order 1 if and only if (i) and (ii) of Theorem 7 hold, just as in the continuous time case [6]. If m=1 (single-input case), K_1=N. Thus (i) of Theorem 7 is the same as (i) of Theorem 3. Since $E_1 = E_1 \cap E$, (ii) of Theorem 7 is trivially satisfied. Since the operator ζ^1 is the same as the operator ζ in the previous section, $\gamma_{\rho-1}^1 = \beta_{\rho-1}$. Since $D_{\rho-1}^1 = A_{\rho-1}$, (iii) of Theorem 7 is the same as (ii) of Theorem 3. Therefore, Theorem 7 is a generalized version of Theorem 3.

Now a sufficient condition for local linearizability is given in the following theorem.

<u>Theorem 8</u>: Suppose that $f(x,u)$ of Σ is an analytic $\mathbb{R}^N$-valued function. Σ is locally linearizable at $(0,0)$ if

(i) the elements of E are linearly independent,

(ii) span E_i = span $(E_i \cap E)$, for $1 \leq i \leq m$, and

(iii) there exists $k(<\infty)$ such that $\gamma_i^\ell \in \text{span}\,((F_i^\ell)^k)$ for $1 \leq \ell \leq m$ and $i \geq 1$, where $(F_i^\ell)^k$ is composed of the first k columns of D_i^ℓ.

Given the system (4) we choose the Kronecker indices $\{K_i\}_{i=1}^m$ in a similar way to the continuous time case (Hunt and Su 1981). First we form the matrix

$$\begin{bmatrix} (\frac{\partial f}{\partial u_1})(0,0) & (\frac{\partial f}{\partial u_2})(0,0) & \cdots & (\frac{\partial f}{\partial u_m})(0,0) \\ (\frac{\partial f}{\partial x})(0,0)(\frac{\partial f}{\partial u_1})(0,0) & (\frac{\partial f}{\partial x})(0,0)(\frac{\partial f}{\partial u_2})(0,0) & \cdots & (\frac{\partial f}{\partial x})(0,0)(\frac{\partial f}{\partial u_m})(0,0) \\ \vdots & \vdots & & \vdots \\ (\frac{\partial f}{\partial x})^{N-1}(0,0)(\frac{\partial f}{\partial u_1})(0,0) & (\frac{\partial f}{\partial x})^{N-1}(0,0)(\frac{\partial f}{\partial u_2})(0,0) & \cdots & (\frac{\partial f}{\partial x})^{N-1}(0,0)(\frac{\partial f}{\partial u_m})(0,0) \end{bmatrix}$$

Set α_i = number of linearly independent vectors in the first $(i+1)$ rows, for $0 \leq i \leq N-1$. Take $\gamma_0 = \alpha_0$ and $\gamma_i = \alpha_i - \alpha_{i-1}$ for $1 \leq i \leq N-1$, and define κ_i to be the number of γ_j with $\gamma_j \geq i$.

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Appendix

Lemma A.1: The equation (2) holds.

Proof: Clearly,

$$D_\xi^m D_u(h\circ\hat{f}^i) = \sum_{\ell=0}^{m} (B^i_{m,\ell})_{(x,u)}(D_x^{\ell+1}h)_{\hat{f}^i(x,u)} \quad \text{and}$$

$$D_\xi^{m+1} D_u(h\circ\hat{f}^i) = \sum_{\ell=0}^{m+1} (B^i_{m+1,\ell})_{(x,u)}(D_x^{\ell+1}h)_{\hat{f}^i(x,u)}$$

where $\{B^i_{m,\ell}\}$ are to be determined. Then

$$B^i_{m+1,0} = D_\xi(B^i_{m,0}) \tag{A.1}$$

$$B^i_{m+1,\ell} = D_\xi(B^i_{m,\ell}) + D_\xi\hat{f}^{i^T} \otimes B^i_{m,\ell-1} \quad \text{for } 1 \le \ell \le m, \tag{A.2}$$

$$B^i_{m+1,m+1} = D_\xi\hat{f}^{i^T} \otimes B^i_{m,m}. \tag{A.3}$$

By (A.1), since $B^i_{0,0} = D_u\hat{f}^{i^T}$, $B^i_{m,0} = D_\xi^m D_u\hat{f}^{i^T}$ for $m \ge 1$. Note that (2) is true when $m=1$. Now suppose that (2) is true for $m \le p$. Let $1 \le \ell \le p$. Then, by (A.2),

$$\begin{aligned} B^i_{p+1,\ell} &= D_\xi(B^i_{p,\ell}) + D_\xi\hat{f}^{i^T} \otimes B^i_{p,\ell-1} \\ &= \sum_{k_1=1}^{p-\ell+1} \sum_{k_2=1}^{k_1} \cdots \sum_{k_\ell=1}^{k_{\ell-1}} D_\xi^{p+1-\ell+1-k_1}(D_\xi\hat{f}^{i^T} \otimes D_\xi^{k_1-k_2}(D_\xi\hat{f}^{i^T} \otimes \cdots \otimes D_\xi^{k_{\ell-1}-k_\ell} \\ &\quad \cdot(D_\xi\hat{f}^{i^T} \otimes D_\xi^{k_\ell-1}D_u\hat{f}^{i^T})\ldots)) + \sum_{k_1=1}^{p+1-\ell+1} \sum_{k_2=1}^{k_1} \cdots \sum_{k_{\ell-1}=1}^{k_{\ell-2}} D_\xi\hat{f}^{i^T} \otimes D^{p+1-\ell+1-k_1} \\ &\quad \cdot(D_\xi\hat{f}^{i^T} \otimes D_\xi^{k_1-k_2}(D_\xi\hat{f}^{i^T} \otimes \cdots \otimes D_\xi^{k_{\ell-2}-k_{\ell-1}}(D_\xi\hat{f}^{i^T} \otimes D_\xi^{k_{\ell-1}-1} D_u\hat{f}^{i^T})\ldots)) \end{aligned}$$

Changing the dummy variables $k_1, k_2, \ldots, k_{\ell-1}$ of the second term into $k_2, k_3, \ldots, k_\ell$, respectively, the second term becomes

$$\sum_{k_2=1}^{p+1-\ell+1} \sum_{k_3=1}^{k_2} \cdots \sum_{k_\ell=1}^{k_{\ell-1}} D_\xi f^{i^T} \otimes D_\xi^{p+1-\ell+1-k_2}(D_\xi f^{i^T} \otimes D_\xi^{k_2-k_3}(D_\xi f^{i^T} \otimes \cdots \otimes D_\xi^{k_{\ell-1}-k_\ell}$$

$$\cdot(D_\xi \hat{f}^{i^T} \otimes D_\xi^{k_\ell-1} D_u \hat{f}^{i^T})\ldots))$$

$$= \sum_{k_1=p+1-\ell+1}^{p+1-\ell+1} \sum_{k_2=1}^{k_1} \sum_{k_3=1}^{k_2} \cdots \sum_{k_\ell=1}^{k_{\ell-1}} D^{p+1-\ell+1-k_1}$$

$$\cdot(D_\xi \hat{f}^{i^T} \otimes D_\xi^{k_1-k_2}(D_\xi \hat{f}^{i^T} \otimes D_\xi^{k_2-k_3}(D_\xi \hat{f}^{i^T} \otimes \cdots \otimes D_\xi^{k_{\ell-1}-k_\ell}(D_\xi \hat{f}^{i^T} \otimes D_\xi^{k_\ell-1} D_u \hat{f}^{i^T})\ldots))$$

Thus (2) is true for $m = p+1$ and $1 \leq \ell \leq p$. By (A.3), it is easy to see that (2) is true for $m = \ell = p+1$. Hence (2) is true for $m = p+1$. By induction, (2) is true for $m \geq 1$.

Let $h(x)$: $\mathbb{R}^N \to \mathbb{R}$ be a C^∞ function.

<u>Lemma A.2</u>: If

$$(S_1\ S_2\ \ldots\ S_k) \begin{bmatrix} (D_x^2 h)_{x=0} \\ (D_x^3 h)_{x=0} \\ \vdots \\ (D_x^{k+1} h)_{x=0} \end{bmatrix} = d_{p\times 1}, \tag{A.4}$$

where S_i is a $p\times N^{i+1}$ matrix for $1 \leq i \leq k$, then $d \in \text{Image}(B)$, where $B = ((\underset{2!}{\otimes} S_1),(\underset{3!}{\otimes} S_2),\ldots,(\underset{(k+1)!}{\otimes} S_k))$.

<u>Proof</u>: (A.4) is equivalent to

$$S_1(D_x^2 h)_{x=0} + S_2(D_x^3 h)_{x=0} + \ldots + S_k(D_x^{k+1} h)_{x=0} = d.$$

Consider

$$S_1(D_x^2 h)_{x=0} = d'.$$

Note that $\left(\frac{\partial^2 h}{\partial x_i \partial x_j}\right)_{x=0} \triangleq h_{ij} = h_{ji} \triangleq \left(\frac{\partial^2 h}{\partial x_j \partial x_i}\right)_{x=0}$ for $1 \le i \le N$ and $1 \le j \le N$. Let $(s_1)_i$ be the i^{th} column of S_1. Then, since $h_{ij} = h_{ji}$,

$$S_1(D_x^2 h)_{x=0} = \sum_{a_1=1}^{N} (s_1)_{(a_1-1)N+a_1} h_{a_1 a_1} + \sum_{a_1=1}^{N} \sum_{a_2=a_1+1}^{N} ((s_1)_{(a_1-1)N+a_2}$$
$$+(s_1)_{(a_2-1)N+a_1}) h_{a_1 a_2} = d'.$$

Therefore, $d' \in$ span (Q), where

$$Q = \left(\bigcup_{a_1=1}^{N} \bigcup_{a_2=a_1+1}^{N} \{s_{(a_1-1)N+a_2} + s_{(a_2-1)N+a_1}\}\right) \cup \left(\bigcup_{a_1=1}^{N} \{s_{(a_1-1)N+a_1}\}\right).$$

Now consider the matrix S_1' defined by

$$S_1' = \underset{2!}{\bullet}\, S_1 = S_1(U_{12} + U_{21}).$$

It is easy to see that the $(a_1-1)N+a_2$-th column of S_1'

$$(s_1')_{(a_1-1)N+a_2} = \begin{cases} 2(s_1)_{(a_1-1)N+a_1}, & \text{if } a_1 = a_2 \\ (s_1)_{(a_1-1)N+a_2} + (s_1)_{(a_2-1)N+a_1}, & \text{if } a_1 \ne a_2 \end{cases}$$

where $1 \le a_1 \le N$ and $1 \le a_2 \le N$. Clearly Image (S_1') = span (Q). Similar arguments can be applied for $(D_x^3 h)_{x=0}, \ldots, (D_x^{k+1} h)_{x=0}$. Therefore, $d \in$ Image (B). (Q.E.D.)

END

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